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An Extension of Geometric Programming with Applications in Engineering Optimization★

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SUMMARY
An extension of geometric programming to handle rational functions of posynomials is presented. The solution technique consists of successive approximations of posynomials and solution of ordinary geometric programs. An example of a multistage heat exchanger system optimization illustrates the computational method.

1. Introduction

This work deals with a new optimization technique motivated by research in the area of optimal engineering design by geometric programming [1, 2, 3, 7, 9, 14]. A serious limitation in the successful application of geometric programming to optimizing engineering design has been that all the functions involved in the problem were required to be posynomials, i.e. generalized polynomials with positive coefficients. Recently, Avriel and Williams [5] extended the theory of geometric programming to include generalized polynomials with unrestricted coefficients. Some aspects of this theory were also treated by Passy and Wilde [13], [14], Blau and Wilde [7], Eben and Ferron [11] and others. The theory developed in [5] is different from all previous attempts to generalize geometric programming. Duffin [10], however, has recently reported that several other papers, in manuscript form at the time of the writing, describe similar approaches. It is based on a method for finding the minimum of an objective function over sets which are the intersections of certain convex sets with complements of convex sets [4]. Such an optimization problem is called complementary convex programming. The present work concerns a method for finding the minimum of a function over sets which can be viewed as intersections of posynomial sets (as they appear in geometric programming) with complements of posynomial sets. Such an optimization problem we call *Complementary Geometric Programming* (CGP).

Most recent textbooks on optimization theory and practice contain sufficient background material on geometric programming for the understanding of this paper, see, e.g. [6, 9, 15, 16]. We shall therefore, repeat here only those aspects of geometric programming which are absolutely necessary for the subject of this paper.

2. Problem Statement

A posynomial P is a real valued function consisting of a finite sum of positive terms, given by

$$P(x) = \sum_{j=1}^{J} c_j \prod_{i=1}^{m} x_i^{a_{ij}} \tag{1}$$

defined for positive values of the vector $x=(x_1, \ldots, x_m)$, where the c_j are *positive* constants and the a_{ij} are arbitrary real constants.

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The primal problem of geometric programming can be written as

$$\text{(PGP)} \qquad \min P_0(x) \tag{2}$$

subject to

$$P_k(x) \leqq 1\,, \qquad k=1, \ldots, K \tag{3}$$

$$x > 0 \tag{4}$$

where the P_k, $k=0, \ldots, K$, are posynomials consisting of $J(k)$ terms. The set of points satisfying (3) and (4) is called a posynomial set.

Associated with the above primal problem is the following dual problem of geometric programming:

$$\text{(DGP)} \qquad \max v(\delta) = \prod_{k=0}^{K} \prod_{j=1}^{J(k)} (c_{jk}/\delta_{jk})^{\delta_{jk}} (\lambda_k)^{\lambda_k} \tag{5}$$

where

$$\lambda_k = \sum_{j=1}^{J(k)} \delta_{jk}\,, \qquad k=0\,, \ldots, K \tag{6}$$

and subject to

$$\sum_{j=1}^{J(0)} \delta_{j0} = 1 \tag{7}$$

$$\sum_{k=0}^{K} \sum_{j=1}^{J(k)} a_{ijk}\delta_{jk} = 0\,, \qquad i=1, \ldots, m \tag{8}$$

$$\delta = (\delta_{10}, \ldots, \delta_{J(K),K}) \geqq 0 \tag{9}$$

A primal geometric program is said to be superconsistent if there exists a vector

$$\hat{x} > 0 \tag{10}$$

such that

$$P_k(\hat{x}) < 1\,, \qquad k=1, \ldots, K \tag{11}$$

The basic duality theorem of geometric programming [9] states then, that if a primal geometric program is superconsistent and has an optimal solution x^*, then its corresponding dual problem also has an optimal solution δ^*. Moreover, the optimal values of the objective functions $P_0(x^*)$ and $v(\delta^*)$ are equal and the primal and dual optimal values are related by

$$\delta^*_{j0} = P_{j0}(x^*)/P_0(x^*) \tag{12}$$

$$\delta^*_{jk} = P_{jk}(x^*)\lambda^*_k \qquad k=1, \ldots, K\,, \quad \lambda^*_k > 0 \tag{13}$$

The above duality theorem has important computational consequences, since it is usually easier to solve the dual problem, which has linear constraints, than the primal problem having nonlinear inequality constraints. Equations (12) and (13) may then be used to obtain the primal optimal solution x^* given the dual variables δ^*. (In general, (12) and (13) are best solved by taking logarithms of both sides and obtaining simultaneous equations, linear in $\log x$).

Primal and dual geometric programs can be easily transformed to convex programming problems by a suitable change of variables and a monotonic transformation. Accordingly, every local optimum of a primal or dual geometric program is also a global optimum. Thus, geometric programs are amenable to analysis and numerical solution by nonlinear convex programming methods.

Suppose now that we do not restrict the c_j in (1) to be positive constants, i.e. they can take on any real value. In this case, a sum of terms as in (1) can be viewed as a difference of two posynomials. Letting R_k, S_k be posynomials for $k=0, \ldots, K$, we can consider then the following problem

$$\text{(QGP)} \qquad \min R_0(x) - S_0(x) \tag{14}$$

subject to

$$R_x(x) - S_k(x) \leqq 1\,, \qquad k = 1, \ldots, K \tag{15}$$

$$x > 0 \tag{16}$$

in which some of the posynomials may be absent. The objective function $R_0(x) - S_0(x)$ is, however, assumed to be positive for all feasible x. This is not a serious restriction since a positive constant can be always added to R_0 to make (14) positive.

In a further generalization of geometric programming we can have rational functions of posynomials in the objective function and constraints, i.e. we can consider

$$\text{(RGP)} \qquad \min \frac{R_0(x) - S_0(x)}{\hat{R}_0(x) - \hat{S}_0(x)} \tag{17}$$

subject to

$$\frac{R_k(x) - S_k(x)}{\hat{R}_k(x) - \hat{S}_k(x)} \leqq 1\,, \qquad k = 1, \ldots, K \tag{18}$$

$$x > 0 \tag{19}$$

where the R_k, S_k, $\hat{R}_k$, $\hat{S}_k$ are all posynomials and the objective function is, again, assumed to be positive for all feasible x. In addition, the denominators of (17) and (18) must not vanish in the feasible region.

By introducing a new variable, x_0, and by elementary algebraic manipulations, it is clear that the QGP and RGP problems can be written as *complementary geometric programs*

$$\text{(CGP)} \qquad \min x_0 \tag{20}$$

subject to

$$\frac{P_k(x)}{Q_k(x)} \leqq 1\,, \qquad k = 0, \ldots, K \tag{21}$$

$$x = (x_0, \ldots, x_m) > 0 \tag{22}$$

where the P_k and Q_k are posynomials.

Complementary geometric programming enables one to handle a much larger family of engineering optimization problems than ordinary geometric programming. This can be attained, however, only by sacrificing certain remarkable properties of geometric programs. Complementary geometric programs can have local minima which are *not* global minima and there is no single transformation to convex programming and no dual program to CGP (in the ordinary sense) can be written.

3. The Algorithm

The algorithm to solve complementary geometric programs is based on the observation that a posynomial divided by a posynomial consisting of only one term is again a posynomial. If, therefore, each of the $Q_k(x)$ in (21) are approximated by one term posynomials, we obtain an ordinary geometric program. The algorithm consists of successively approximating the $Q_k(x)$ by one term posynomials so as to produce a sequence of approximating geometric programs whose solutions converge to a local minimum of the given CGP program.

The approximation is based on the *arithmetic-geometric inequality*

$$\Sigma_j \mu_j \geqq \Pi_j [\mu_j/\delta_j]^{\delta_j} \tag{23}$$

which holds for any positive numbers μ_j and any non-negative numbers δ_j such that $\Sigma\, \delta_j = 1$. Since the $Q_k(x)$ are of the form (omitting for the moment the k subscript)

$$Q(x) = \Sigma_j q_j(x) \; ; \qquad q_j(x) = c_j \Pi_i x_i^{a_{ij}}$$

we may take any $x > 0$, and put

$$\mu_j = q_j(x) \; ; \qquad \delta_j = \frac{q_j(\bar{x})}{Q_j(\bar{x})} \, .$$

From (23), therefore

$$Q(x) \geqq \Pi_j \left(\frac{Q(\bar{x})}{q_j(\bar{x})} \, q_j(x) \right)^{q_j(\bar{x})/Q(\bar{x})} \tag{24}$$

The right hand side of (24) is a one term posynomial; it is the *approximation* for $Q(x)$ *at* $\bar{x}$ and will be denoted by $Q(x, \bar{x})$.

The first step in solving a complementary geometric program then is to select some feasible point, call it $x^{(1)}$, and replace the $Q_k(x)$ by $Q_k(x, x^{(1)})$. Thus (20), (21), (22) become

$$\min x_0 \tag{25}$$

subject to

$$\frac{P_k(x)}{Q_k(x, x^{(1)})} \leqq 1 \, , \qquad k = 0, 1, \ldots, K \tag{26}$$

$$x > 0 \tag{27}$$

This *ordinary* geometric program is solved for some optimal solution; call it $x^{(2)}$. $Q(x)$ is then replaced by $Q(x, x^{(2)})$ and a new optimal solution $x^{(3)}$ is obtained, etc. Note that, by (24), if $x^{(1)}$ is feasible, than so is $x^{(2)}$, since

$$1 \geqq \frac{P_k(x^{(2)})}{Q_k(x^{(2)}, x^{(1)})} \geqq \frac{P_k(x^{(2)})}{Q_k(x^{(2)})}$$

The sequence $x^{(\alpha)}$, therefore, is feasible. Subject to mild regularity conditions (see [5] for details) this sequence will converge to a local minimum of the complementary geometric program.

An interesting computational aspect of the above algorithm is the "degree of difficulty" of a complementary geometric programming problem.

The notion of "degree of difficulty" of an ordinary geometric program was introduced in [9]. For a given primal (or dual) geometric program the degree of difficulty is equal to the total number of posynomial terms (equals number of dual variables) appearing in the problem less $(m+1)$, where m is the number of primal variables (orthogonality constants (8)). The degree of difficulty is essentially equal to the number of independent variables over which the dual objective function is to be maximized, subject to the nonnegativity constraints of all the dual variables. Well formulated geometric programs always have a nonnegative degree of difficulty. For complementary geometric programs the degree of difficulty is defined [5] as the total number of posynomial terms in the *numerators* of (21) less $(m+1)$, where $m+1$ is the number of primal variables $x_0, \ldots, x_m$.

In other words, the degree of difficulty of complementary geometric programs is equal to the degree of difficulty of the approximating ordinary geometric programs, solved at each iteration. This means, then, that the degree of difficulty of a CGP problem is independent of the number of terms appearing in the denominators of constraints (21). Thus a CGP problem with primal variables $x_0, x_1, \ldots, x_m$ and a total number of m posynomial terms in the *numerators* of the constraints

$$\frac{P_k(x)}{Q_k(x)} \leqq 1 , \qquad k=0, \ldots, K \tag{28}$$

has zero degrees of difficulty and at each iteration we only solve a square system of non-homogeneous linear equations.

The following example illustrates the notions introduced in this paper by a simple example. The second example demonstrates applications of complementary geometric programming to engineering design problems.

Example 1. A Complementary Geometric Program with Zero Degrees of Difficulty

To illustrate the method of solution of complementary geometric programs consider the following problem:

$$\min x_0 \tag{29}$$

$$8x_0^2+8x_1 \geqq 11 \tag{30}$$

$$-x_0+8x_1 \leqq 2 \tag{31}$$

$$x_0>0 , \quad x_1>0 \tag{32}$$

First, we rearrange (30) and (31) to bring them into the form of constraints (21). We obtain

$$\frac{P_0(x)}{Q_0(x)} = \frac{11/8}{x_0^2+x_1} \leqq 1 \tag{33}$$

$$\frac{P_1(x)}{Q_1(x)} = \frac{8x_1}{2+x_0} \leqq 1 \tag{34}$$

and our CGP problem is to minimize x_0 subject to constraints (32), (33) and (34).

We first approximate the denominators of (33) and (34) by the formula (24). For any positive $\bar{x}=(\bar{x}_0, \bar{x}_1)$ we have

$$Q_0(x, \bar{x}) = \left(\frac{\bar{x}_0^2+\bar{x}_1}{\bar{x}_0^2} x_0^2\right)^{\bar{x}_0/(\bar{x}_0^2+\bar{x}_1)} \left(\frac{\bar{x}_0^2+\bar{x}_1}{\bar{x}_1} x_1\right)^{\bar{x}_1/(\bar{x}_0^2+\bar{x}_1)} \tag{35}$$

$$Q_1(x, \bar{x}) = \left(\frac{2+\bar{x}_0}{2} 2\right)^{2/(2+\bar{x}_0)} \left(\frac{2+\bar{x}_0}{\bar{x}_0} x_0\right)^{\bar{x}_0/(2+\bar{x}_0)} \tag{36}$$

Suppose we start at the feasible point $x^{(1)}=(4, \frac{1}{4})$. Then

$$Q_0(x, x^{(1)}) = 65 (\tfrac{1}{4})^{192/65} (x_0)^{128/65} (x_1)^{1/65} \tag{37}$$

$$Q_1(x, x^{(1)}) = 6(4)^{-2/3} (x_0)^{2/3} \tag{38}$$

so that the first approximating geometric program is

$$\min x_0 \tag{39}$$

subject to

$$x_0>0 \quad x_1>0 \tag{40}$$

$$\left[\frac{11}{130} (4)^{127/65}\right] x_0^{-128/65} x_1^{-1/65} \leqq 1 \tag{41}$$

$$[\tfrac{1}{3}(4)^{5/3}] x_0^{-2/3} x_1 \leqq 1 \tag{42}$$

This program has a total number of three posynomial terms and two primal variables, hence it has zero degrees of difficulty. Its solution can be easily found by solving the following set of linear equations (dual constraints)

$$\begin{aligned} &\delta_1 = 1 \\ &\delta_1 - 1/65\,\delta_2 + \delta_3 = 0 \\ &\delta_1 - 128/65\,\delta_2 - 2/3\,\delta_3 = 0 \end{aligned} \tag{43}$$

which has the unique solution $\delta_1^{(1)}=1$, $\delta_2^{(1)}=195/386$ and $\delta_3^{(1)}=3/386$.

Substituting these dual variables into the dual objective function we get

$$v(\delta^{(1)}) \cong 1.139 \tag{44}$$

By solving equations (12) and (13) we obtain for the optimal variables of the first approximating geometric program $x_0 \simeq 1.139$ and $x_1 \simeq 0.325$. Next we choose $x^{(2)}=(1.139, 0.325)$, and re-approximate Q_0 and Q_1 around this point, formulate and solve a new ordinary geometric program, etc. The following table illustrates the convergence of the algorithm to the desired minimum.

TABLE I

Convergence to Optimum in Example 1.

Iteration	x_0	x_1
0	4.000	0.250
1	1.139	0.325
2	1.009	0.375
3	1.000	0.375

Example 2. Multistage Heat Exchanger Design by Complementary Geometric Programming

Complementary geometric programming will now be applied to a simple three-stage heat exchanger design problem, solved previously by Boas by dynamic programming [8] and also by Fan and Wang via the maximum principle [12]. Our purpose here is to demonstrate the method of complementary geometric programming by a simple example without comparing its efficiency to other well-known optimization techniques. Consider then a system of three heat exchangers as illustrated in Figure 1.

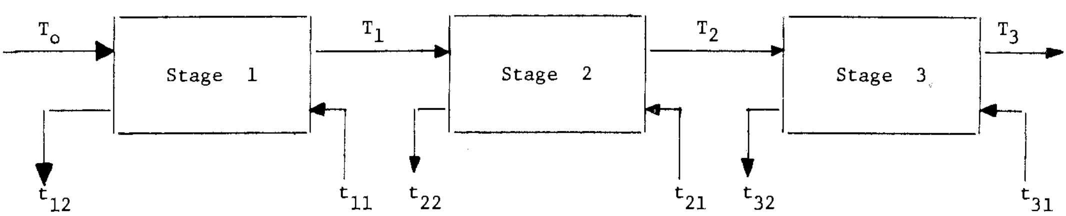

Figure 1. Three Stage Heat Exchanger System.

A fluid having a given flow rate W and specific heat C_p is heated from temperature T_0 to T_3 by passing three heat exchangers in series. In each heat exchanger (stage) the cold stream is heated by a hot fluid having the same flow rate W and specific heat C_p as the cold stream. The temperatures of the hot fluid entering the heat exchangers, t_{11}, t_{21} and t_{31} and the overall heat transfer coefficients U_1, U_2, U_3 of the exchangers are known constants. Optimal design involves minimizing the sum of the heat transfer areas of the three exchangers, $A_T = A_1 + A_2 + A_3$. As in [3] we write all the design relations as inequalities rather than equations, the sense of the inequalities being determined by physical reasons.

There are three heat balances expressing the fact that the rate of heat transferred to the cold fluid is less than or equal to the rate of heat lost by the hot stream:

$$WC_p(T_i - T_{i-1}) \leqq WC_p(t_{i1} - t_{i2}), \qquad i = 1, 2, 3 \tag{45}$$

or

$$T_i + t_{i2} \leqq t_{i1} + T_{i-1}, \qquad i = 1, 2, 3 \tag{46}$$

Similarly, we can write three heat transfer inequalities:

$$WC_p(T_i - T_{i-1}) \leqq U_i A_i (t_{i2} - T_{i-1}), \qquad i = 1, 2, 3 \tag{47}$$

Rearranging (46) and (47) we get the following complementary geometric program:

$$\min A_T \tag{48}$$

subject to

$$\frac{A_1 + A_2 + A_3}{A_T} \leqq 1 \tag{49}$$

$$\frac{T_1 + t_{12}}{t_{11} + T_0} \leqq 1 \tag{50}$$

$$\frac{T_2 + t_{22}}{t_{21} + T_1} \leqq 1 \tag{51}$$

$$\frac{T_3 + t_{32}}{t_{31} + T_2} \leqq 1 \tag{52}$$

$$\frac{T_1 + \hat{U}_1 A_1 T_0}{T_0 + \hat{U}_1 A_1 t_{12}} \leqq 1 \tag{53}$$

$$\frac{T_2 + \hat{U}_2 A_2 T_1}{T_1 + \hat{U}_2 A_2 t_{22}} \leqq 1 \tag{54}$$

$$\frac{T_3 + \hat{U}_3 A_3 T_2}{T_2 + \hat{U}_3 A_3 t_{32}} \leqq 1 \tag{55}$$

$$A_i > 0, \quad t_{i2} > 0, \quad i = 1, 2, 3; \qquad T_i > 0, \quad i = 1, 2; \quad A_T > 0 \tag{56}$$

where

$$\hat{U}_i = U_i / WC_p, \qquad i = i, 2, 3. \tag{57}$$

This program has $15 - 9 = 6$ degrees of difficulty.

For a numerical solution of the above program we used the same data as in [8] and [12]:

$$T_0 = 100°\text{F}, \quad T_3 = 500°\text{F}, \quad WC_p = 10^5 \text{ (B.t.u./hr} - °\text{F)}, \quad \text{and}$$

i	T_{i1} (°F)	U_i(B.t.u./hr – sq. ft. – °F)
1	300	120
2	400	80
3	600	40

It can be seen in the following table that starting with a guess of $A_T = 15{,}000$ we attained convergence in four iterations to a minimum total heat transfer area of approximately 7049 sq.ft., similar to the values reported in [8] and [12].

TABLE II

Solution of Three Stage Heat Exchanger Design Problem

Iteration	A_T	A_1	A_2	A_3	T_1	T_2	t_{12}	t_{22}	t_{32}
0	15000	5000	5000	5000	200	350	150	225	425
1	7664	783	2044	4837	176	308	224	268	407
2	7120	599	1590	4931	182	303	218	280	403
3	7049	579	1370	5100	182	296	218	286	396
4	7049	567	1357	5125	181	295	219	286	395

This solution was obtained by an experimental computer code written for complementary geometric programs by M. Rusin and G. Crane at the Central Research Laboratories of Mobil Research and Development Corp., Princeton, New Jersey.

Conclusions

Complementary geometric programming offers a significant extension of the applicability of geometric programming to engineering optimization problems. An algorithm for the solution of complementary geometric programming problems by successive solutions of certain approximating ordinary geometric programs was presented. Thus complementary geometric programs can be solved by existing computer codes for geometric programming after introducing some minor modifications. The degree of difficulty of each of the approximating programs is equal to the number of posynomial terms appearing in the numerators of the constraints of a CGP problem less the number of variables in the problem and is independent of the number of terms in the denominators of the constraints. In case of a QGP problem, i.e. a geometric programming type problem with positive and negative terms, this is equivalent to saying that the degree of difficulty is independent of the number of negative terms.

Application of CGP to engineering design problems was demonstrated by finding the minimum total heat transfer area of a three-stage heat exchanger system. The solution obtained in four iterations verified previous solutions of the same problem.

REFERENCES

[1] M. Avriel, Fundamentals of Geometric Programming, in *Applications of Mathematical Programming, Techniques*, Beale, E. M. L., (ed.), The English Universities Press, London, 1970.

[2] M. Avriel and D. J. Wilde, Optimal Condenser Design by Geometric Programming, *Ind. Eng. Chem. Process Design and Dev.*, 6 (1967) 256–263.

[3] M. Avriel and D. J. Wilde, Engineering Design Under Uncertainty, *Ind. Eng. Chem. Process Design and Dev.*, 8 (1969) 124–131.

[4] M. Avriel and A. C. Williams, Complementary Convex Programming, Mobil R&D Corp. Progress Memorandum, May 1968 (unpublished).

[5] M. Avriel and A. C. Williams, Complementary Geometric Programming, *SIAM J. Appl. Math.*, 19 (1970) 125–141.

[6] S. G. Beveridge and R. S. Schechter, *Optimization: Theory and Practice*, McGraw-Hill, New York, 1970.

[7] G. E. Blau and D. J. Wilde, Generalized Polynomial Programming, *Can. J. Chem. Eng.*, 47 (1969) 317–326.

[8] A. H. Boas, Optimization via Linear and Dynamic Programming, *Chem. Eng.*, 70 April 1, 1963, 85–88.

[9] R. J. Duffin, E. L. Peterson and C. Zener, *Geometric Programming*, Wiley, New York, 1967.

[10] R. J. Duffin, Linearizing Geometric Programs, *SIAM Rev.*, 12, 2 (1970) 211–237.

[11] C. D. Eben and J. R. Ferron, A Conjugate Inequality for General Means with Applications to Extremum Problems, *A.I.Ch.E. J.*, 14 (1968) 32–37.

[12] L. T. Fan and C. S. Wang, *The Discrete Maximum Principle*, Wiley New York, 1964.

[13] U. Passy and D. J. Wilde, Generalized Polynomial Optimization, *SIAM J. Appl. Math.*, 15 (1967) 1344–1356.

[14] U. Passy and D. J. Wilde, Mass Action and Polynomial Optimization, *J. of Eng. Math.*, 3 (1969) 325–335.

[15] D. J. Wilde and C. S. Beightler, *Foundations of Optimization*, Prentice-Hall, Englewood Cliffs, 1967.

[16] W. I. Zangwill, *Nonlinear Programming: A Unified Approach*, Prentice-Hall, Englewood Cliffs, 1969.